Vorticity – Stream function Solver in Cylindrical Coordinates

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This document summarizes equations used to solve flow in a cylindrical pipe using the stream function approach as seen in `https://www.particleincell.com/2016/vorticity-streamfunction-cylindrical`.

1 Governing Equations

1.1 Velocity Components

Velocity components of an incompressible axisymmetric flow can be described using Stokes stream function ψ as [1]

$$\begin{aligned} u_z = u &= \frac{1}{r}\frac{\partial \psi}{\partial r} \\ u_r = v &= -\frac{1}{r}\frac{\partial \psi}{\partial z} \end{aligned} \tag{1}$$

The azimuthal component u_θ does not depend on the stream function and can be defined independently. In this writeup, $u_\theta = 0$. Also, the volumetric flow bounded by streamtube ψ is $Q = 2\pi\psi$. This formulation automatically satisfies continuity $\nabla \cdot \vec{u} = 0$, since

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_z}{\partial z} = -\frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial z} + \frac{1}{r}\frac{\partial^2 \psi}{\partial z \partial r} = 0$$

1.2 Vorticity

Vorticity is defined as $\vec{\omega} = \nabla \times \vec{v}$. For axisymmetric flow with $\partial/\partial\theta = 0$ and $u_\theta = 0$ only the w_θ component survives and we have

$$\omega = \omega_\theta = \frac{\partial v}{\partial z} - \frac{\partial u}{\partial r} \tag{2}$$

1.3 Stream function governing equation

By substituting definitions of velocity in terms of Stokes stream function, Eq. 1, into Eq. 2 for vorticity, we obtain

$$\begin{aligned} \omega &= -\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial z}\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) \\ &= -\frac{1}{r}\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial \psi}{\partial r} \end{aligned}$$

or

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = -\omega r \tag{3}$$

The left side of this equation is not $\nabla^2 \psi$ due to the negative sign on the last term on the left side.

1.4 Vorticity Transport Equation

The temporal evolution of vorticity is given by the vorticity transport equation. This equation is normally derived by taking curl of the momentum equation, for instance see [2] for details, and is given by

$$\frac{\partial \omega}{\partial t} + \vec{v} \cdot \nabla \omega = \nu \nabla^2 \omega$$

For the axisymmetric flow we have

$$\frac{\partial \omega}{\partial t} + u\frac{\partial w}{\partial z} + v\frac{\partial w}{\partial r} = \nu \left[\frac{\partial^2 \omega}{\partial z^2} + \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} \right] \tag{4}$$

2 Finite Difference Form

2.1 Stream function

Equation 3 is discretized using standard central difference as

$$\frac{1}{\Delta^2 z}(\psi_{i-1,j} - 2\psi_{i,j} + \psi_{i+1,j}) + \frac{1}{\Delta^2 r}(\psi_{i,j-1} - 2\psi_{i,j} + \psi_{i,j+1}) - \frac{1}{2r_{i,j}\Delta r}(\psi_{i,j+1} - \psi_{i,j-1}) = \omega_{i,j} r_{i,j} \tag{5}$$

This equation is solved using an SOR-accelerated Jacobi solver, with convergence check $||\psi^{k+1} - \psi^k|| < \epsilon_{tol}$.

2.2 Vorticity

The vorticity transport equation, Equation 4, is advanced using the Runge-Kutta fourth-order (RK4) method. Letting vorticity transport equation be given by

$$\begin{aligned}
\frac{\partial \omega}{\partial t} &= \nu \left[\frac{\partial^2 \omega}{\partial z^2} + \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} \right] - u\frac{\partial w}{\partial z} - v\frac{\partial w}{\partial r} \\
\frac{\partial \omega}{\partial t} &= R(\omega)
\end{aligned}$$

we have [3]

$$w^{(1)} = w^k + \frac{\Delta t}{2} R^k \tag{6}$$

$$w^{(2)} = w^k + \frac{\Delta t}{2} R^{(1)} \tag{7}$$

$$w^{(3)} = w^k + \Delta t R^{(2)} \tag{8}$$

$$w^{k+1} = w^k + \frac{\Delta t}{6}\left(R^k + 2R^{(1)} + 2R^{(2)} + R^{(3)} \right) \tag{9}$$

3 Boundary Conditions

In order to solve this equation (using finite difference method) we need to specify boundary conditions. There are five types of boundaries to consider for the tube problem: 1) wall, 2) axis of revolution, 3) inlet, 4) outlet on zmax, and 5) outlet on rmax. These are sketched in Figure 3.

3.1 Inlet

We assume the flow entering through the inlet is parallel to the cylinder axis. Thus at the inlet $u_r = v = 0$ or

$$\left.\frac{\partial \psi}{\partial z}\right|_{inlet} = 0 \tag{10}$$

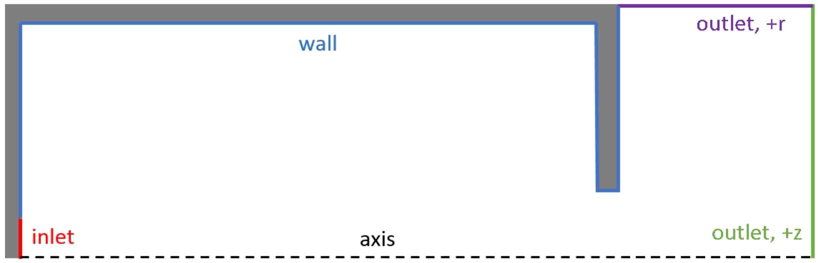

Figure 1: Boundary types for pipe problem

Substituting $v = 0$ into the vorticity equation gives us

$$\omega\big|_{inlet} = -\frac{\partial u}{\partial r} \tag{11}$$

These equations are discretized using first order scheme as $\psi_{0,j} = \psi_{1,j}$ and $\omega_{0,j} = (u_{0,j-1} - u_{0,j+1})/(2\Delta r)$.

3.2 Axis of Revolution

We require $v = 0$ as there can be no flow across the axis of revolution. Therefore $\partial\psi/\partial z = 0$, and value of ψ is constant along the axis. We set this value to zero, giving us

$$\psi\big|_{axis} = 0 \tag{12}$$

Zero radial velocity also implies that along the axis $\omega = -\partial u/\partial r$. Axial symmetry implies $\partial()/\partial r = 0$ at $r = 0$, and

$$\omega\big|_{axis} = 0 \tag{13}$$

3.3 Wall

$Q = 2\pi(\psi_2 - \psi_1)$ is the volumetric flow rate between two stream tubes. With $\psi_1 = 0$ being the axis of revolution, we have Dirichlet

$$\psi\big|_{wall} = \frac{1}{2} u_0 r_{inlet}^2 \tag{14}$$

along the outer wall. Vorticity boundaries along the wall are derived using similar approach to [2]. Since the stream function is constant along a wall, derivatives of ψ in Equation 3 vanish in the wall direction. Along a left wall we have

$$\frac{\partial^2 \psi}{\partial z^2}\bigg|_{wall} = -\omega r$$

Assuming the wall is at $i = L$, we can write the following Taylor series expansion

$$\psi_{L+1,j} = \psi_{L,j} + \frac{\partial \psi}{\partial z}\bigg|_{L,j} \Delta z + \frac{\partial^2 \psi}{\partial z^2}\bigg|_{L,j} \frac{\Delta z^2}{2}$$

Since $\partial\psi/\partial z = -vr$

$$\psi_{L+1,j} = \psi_{L,j} - v_{L,j} r \Delta z + \frac{\partial^2 \psi}{\partial z^2}\bigg|_{L,j} \frac{\Delta z^2}{2}$$

or

$$\frac{\partial^2 \psi}{\partial z^2}\bigg|_{L,j} = \frac{2(\psi_{L+1,j} - \psi_{L,j})}{\Delta z^2} + \frac{2 v_{L,j} r}{\Delta z}$$

and finally

$$\omega\big|_{L,j} = \frac{2(\psi_{L,j} - \psi_{L+1,j})}{r \Delta z^2} - \frac{2 v_{L,j}}{\Delta z} \tag{15}$$

Using similar approach, the boundary condition for a right wall at $i = R$ is found to be

$$\omega\big|_{R,j} = \frac{2(\psi_{R,j} - \psi_{R-1,j})}{r\Delta z^2} + \frac{2v_{R,j}}{\Delta z} \tag{16}$$

Along the top wall, $\partial\psi/\partial z = 0$ and Equation 3 reduces to

$$\left[\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right]_{wall} = -\omega r$$

Again we start by expanding the second derivative,

$$\psi_{i,T-1} = \psi_{i,T} - \frac{\partial\psi}{\partial r}\Big|_{i,T}\Delta r + \frac{\partial^2\psi}{\partial r^2}\Big|_{i,T}\frac{\Delta r^2}{2}$$

Using $\partial\psi/\partial r = ur$, the above reduces to

$$\frac{\partial^2\psi}{\partial r^2}\Big|_{i,T} = \frac{2(\psi_{i,T-1} - \psi_{i,T}}{\Delta r^2} + \frac{2u_{T,j}r}{\Delta r}$$

Substituting into the original equation,

$$\frac{2(\psi_{i,T-1} - \psi_{i,T})}{\Delta r^2} + \frac{2u_{i,T}r}{\Delta r} - u_{i,T} = -\omega r$$

or

$$\omega\big|_{i,T} = \frac{2(\psi_{i,T} - \psi_{i,T-1})}{r\Delta r^2} - \frac{2u_{i,T}}{\Delta r} + \frac{u_{i,T}}{r} \tag{17}$$

There is no bottom wall in this problem, but for generality, the matching boundary condition can be found to be

$$\omega\big|_{i,B} = \frac{2(\psi_{i,B} - \psi_{i,B+1})}{r\Delta r^2} + \frac{2u_{i,B}}{\Delta r} + \frac{u_{i,B}}{r} \tag{18}$$

3.4 Zmax outlet

In general, the flow will be aligned with the z-axis, however, there may be some non-zero v component due to jet expansion. As such, simply setting $\partial\psi/\partial z|_{zmax} = 0$ may not be valid. Following approach in [2], on zmax we let

$$\frac{\partial\psi}{\partial z}\Big|_{zmax} = -vr \tag{19}$$

which is differenced as $\psi_{ni-1,j} = \psi_{ni-2,j} - \Delta z v_{ni-1,j} r_j$. Vorticity boundary condition on the outlet is set as

$$\frac{\partial\omega}{\partial z}\Big|_{zmax} = 0 \tag{20}$$

or $\omega_{ni-1,j} = \omega_{ni-2,j}$

3.5 Rmax outlet

This is the trickiest of all boundaries and I am not particularly sure what boundary condition is most applicable. Generally, we expect there to be very little / no flow here. Setting no-flow boundary is analogous to making this boundary a wall, with $\psi = \psi_{wall}$ and ω set from Equation 17. However, I think more appropriate boundary may be requiring that any flow there may be is perpendicular to the wall, hence $u = 0$ and

$$\frac{\partial\psi}{\partial r}\Big|_{rmax} = 0 \tag{21}$$

which is differenced as $\psi_{i,nj-1} = \psi_{i,nj-2}$. Vorticity boundary condition is set similarly to the zmax outlet

$$\frac{\partial\omega}{\partial r}\Big|_{rmax} = 0 \tag{22}$$

or $\omega_{ni-1,j} = \omega_{ni-2,j}$

References

[1] Wikipedia, "Stokes Stream Function", Accessed May 10th, 2016, https://en.wikipedia.org/wiki/Stokes_stream_function

[2] Salih, A., "Streamfunction-Vorticity Formulation", Department of Aerospace Engineering Indian Institute of Space Science and Technology, March 2013, https://www.iist.ac.in/sites/default/files/people/psi-omega.pdf

[3] Tannehill, J., Anderson, D., Pletcher, R., *Computational Fluid Mechanics and Heat Transfer*, Taylor & Francis, 2nd ed., 1997